Final Exam — Advanced Algebraic Structures (WBMA16000)

Wednesday January 29, 2019, 15:00h–18.00h

University of Groningen

Instructions

1. Write your name and student number on every page you hand in.
2. All answers need to be accompanied with an explanation or a calculation.
3. You may use results obtained in homework or tutorial problems.
4. In total you can obtain at most 90 points on this exam. Your final grade is $(P + 10)/10$, where $P \leq 90$ is the number of points you obtain on the exam.

Problem 1 (5+5 points) (Module Homomorphisms)

(a) Show that $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z})$ is trivial.

(b) Let R be a commutative ring and let $n \geq 1$ be an integer. Show that $\mathrm{Hom}_R(R^n, R) \cong R^n$.

Problem 2 (5+4+6+5 points) (Tensor products)

(a) Find a nontrivial $\mathbb{Z}$-module M such that $M \otimes_{\mathbb{Z}} M \cong M$ and $M \ncong \mathbb{Z}$.

(b) Let R be a commutative ring, let I be an ideal of R and let M be an R-module. Then

$$IM = \left\{ \sum_{i=1}^{n} a_i m_i : n \geq 0, a_i \in I, m_i \in M \text{ for all } i \right\}$$

is a submodule of M (you do not need to prove this). Show that there is a unique R-module-homomorphism

$$f\colon (R/I) \otimes_R M \to M/IM$$

such that $f((r + I) \otimes m) = (rm) + IM$ for all $r + I \in R/I$ and $m \in M$.

(c) Show that f in (b) is an isomorphism. (Hint: Find the inverse function.)

(d) Find an example of a commutative ring R, an ideal I of R and an R-module M such that $I \otimes_R M \ncong IM$.

Problem 3 (5+4+6 points) (Projective modules)

(a) Let $n > 1$ be an integer. Show that the $\mathbb{Z}$-module $\mathbb{Z}/n\mathbb{Z}$ is not projective.

(b) Deduce that a finitely generated $\mathbb{Z}$-module is projective if and only if it's free.

(c) Let p be a prime, let $n \geq 1$ be an integer and let R be the ring $\mathbb{Z}/p^n\mathbb{Z}$. Show that the following property holds for R if and only if $n = 1$: Every submodule of a projective R-module is itself projective.

Problem 4 (3+6+6+6 points) (Cyclotomic and cyclic extensions)

For a positive integer n, let $\Phi_n(x) \in \mathbb{Q}[x]$ be the n-th cyclotomic polynomial over $\mathbb{Q}$ and let $\zeta_n = e^{2\pi i/n} \in \mathbb{C}$.

(a) Write down $\Phi_n(x) \in \mathbb{Q}[x]$ for $n = 7$ and $n = 17$.

(b) For each $n \in \{7, 17\}$ prove that

 (i) there exists $a_n \in \mathbb{Q}$ and $b_n \in \mathbb{Q}(\zeta_n) \setminus \mathbb{Q}$ such that $b_n^2 = a_n$;

 (ii) if $a'_n \in \mathbb{Q}$, $b'_n \in \mathbb{Q}(\zeta_n) \setminus \mathbb{Q}$ satisfy $b'^2_n = a'_n$, then $a'_n = \lambda^2 a_n$ for some $\lambda \in \mathbb{Q}$.

(c) Prove that there exists $f(x) \in \mathbb{Q}[x]$ such that $f(\cos(2\pi/17)) = b_{17}$, but there exists no $g(x) \in \mathbb{Q}[x]$ such that $g(\cos(2\pi/7)) = b_7$.

(d) Give an example of a cyclic extension of $\mathbb{Q}(\zeta_7)$ of degree 7 and an example of a cyclic extension of $\mathbb{Q}(\zeta_{17})$ of degree 17.

Problem 5 (6+6+6+6 points) (Galois group of the splitting field of a cubic)

Let K be a field of characteristic different from 2 and 3 and consider a separable polynomial

$$f(x) = x^3 + ax^2 + bx + c \in K[x].$$

Let L be the splitting field of f over K and let $G = \mathrm{Gal}(L/K)$.

(a) Show that G is isomorphic to a subgroup of S_3.

(b) Assume now that $f(x)$ is irreducible in $K[x]$; deduce that $G \cong A_3$ or $G \cong S_3$. Let $\alpha_1, \alpha_2, \alpha_3 \in L$ be the roots of $f(x)$. Define

$$\Delta = (\alpha_1 - \alpha_2)^2(\alpha_1 - \alpha_3)^2(\alpha_2 - \alpha_3)^2.$$

 (i) Prove that $\Delta \in K$.

 (ii) Prove that Δ is a square in K if and only if $G \cong A_3$.

(c) Let $K = \mathbb{F}_5$. Show that for every irreducible $f(x) \in K[x]$ as above, Δ is a square.

(d) Let K be the splitting field of $x^3 - 5 \in \mathbb{Q}[x]$ and let L be the splitting field of $f(x) = x^3 - 7 \in K[x]$ over K. Prove that $G \cong A_3$.

End of test (90 points)